The famous result of Gibbard and Satterthwaite shows that every voting procedure is manipulable if the voters can have any preferences over the candidates. That is, a voter may improve the voting result by not voting according to his true preference. Approval voting, introduced by Brams and Fishburn, is not manipulable if preferences are dichotomous: each voter only distinguishes between acceptable and non-acceptable candidates. Approval voting offers a compromise between flexibility and non-manipulability of the voting procedure. Based on recent and ongoing research we discuss the extent to which approval voting is manipulable if preferences are more refined. We also provide some evidence that \(k\)-approval voting, in which voters approve of exactly \(k\) candidates, may offer an alternative to approval voting that is better in terms of potential manipulation.

### Strategic manipulation of votes

National elections, the Eurovision Song Festival, and councils of scientific communities have in common that voters choose from candidates by some fixed voting procedure. The theorem of Gibbard (1973) and Satterthwaite (1975) says that if we want such a voting procedure to be non-dictatorial – and dictators are generally disliked – then it will necessarily be strategically manipulable. This means that there are situations in which some voter may obtain a better result by not voting according to his true preference over the candidates. Consider the following example with three voters and five candidates.

<table>
<thead>
<tr>
<th>Voter</th>
<th>(a_1)</th>
<th>(a_2)</th>
<th>(a_3)</th>
<th>(a_4)</th>
<th>(a_5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
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<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

The voters are 1, 2, and 3, and the candidates \(a_1,\ldots,a_5\). The numbers represent preferences. E.g., voter 1 likes \(a_1\) best and \(a_2\) least. These numbers can also be used for voting: \(a_1\) obtains a total score of 9, \(a_2\) of 7, \(a_3\) of 11, \(a_4\) of 8, and \(a_5\) of 10. This particular voting procedure, the Borda rule, therefore results in the social ranking \(a_3, a_5, a_1, a_4, a_2\). If exactly one candidate is to be elected, then this would be candidate \(a_3\). Since preferences are private information, voter 1 could change his scores to 5, 3, 1, 2, 4, for \(a_1,\ldots,a_5\), respectively, resulting in total scores of 9, 9, 9, 8, 10 and thus in \(a_5\) as the final winner. Since voter 1 prefers \(a_5\) over \(a_3\), this voter gains by strategic manipulation.

### Can strategic manipulation be avoided?

Strategic manipulation may result in the ‘wrong’ candidate being elected. In the example above, \(a_3\) seems to be a good compromise but \(a_5\) is the worst candidate for voter 3. The possibility of strategic manipulation may lead to an election result that does not properly reflect the true preferences of the voters. Unfortunately, the Gibbard-Satterthwaite theorem is quite robust and holds whenever there are at least two voters, three candidates, and each individual preference over the candidates is possible. The last condition is crucial. Suppose, for instance, that every profile of preferences is single-peaked. This means that the candidates can be lined up such that each voter’s preference decreases both to the left and the right from his top candidate. As a concrete example, suppose that the five candidates above are the temperatures 18\(^\circ\)-22\(^\circ\) in a room, which can be adjusted by a thermostat. The three inhabitants of the room vote for the temperature. It is reasonable to assume that each person has an ideal temperature and that preference decreases further away from this ideal temperature. Such a profile of preferences is single-peaked. (The reader may want to verify that the preferences...
in the example above are not single-peaked.) Consider the voting procedure that picks the median of the reported ideal temperatures. It is not hard to check that under this procedure no voter can improve the result by strategic voting, i.e., by not reporting his true preference.

Dichotomous preferences and approval voting

We now focus on a different preference restriction, related to the procedure of approval voting as proposed in Brams and Fishburn (1983). Under approval voting, each voter votes for a subset of the candidates, that is, assigns 1 to each of these candidates and 0 to the others. The candidates with most votes are the winners. So this voting procedure results in a set of candidates. (If a fixed number of candidates is to be elected then we need an additional rule but this point is ignored here for simplicity.) A voter’s preference is dichotomous if this voter has a set of top candidates between which he is indifferent, and which he prefers to the other candidates between all of which he is again indifferent. Under quite natural assumptions on how a preference over candidates is extended to a preference over sets of candidates, approval voting cannot be strategically manipulated when preferences are dichotomous. That is, no voter can bring about a better set of candidates by not voting exactly for his own top set. This is a very attractive property but the assumption of dichotomous preferences is quite strong. If we view a voter’s top candidates as his acceptable candidates and the others as non-acceptable, then it is likely that he is not (completely) indifferent between the candidates that he finds acceptable, nor between those that are not acceptable. Then, what does this imply for the strategic manipulability of approval voting?

Strategic manipulability of approval voting

If preferences are not dichotomous then strategic manipulation of approval voting is possible (Brams and Fishburn (1983)). In Peters, Roy and Storcken (2009) we try to obtain some insight into the seriousness of this problem. For the purpose of this article, assume that voters have strict preferences (no indifferences), and a specific top set of candidates that they find acceptable. Strategic manipulation means that a voter can improve the set of winners by not voting exactly for his set of acceptable candidates. To evaluate when a set of winners becomes better we consider three different extensions of strict preferences over candidates to preferences over sets of candidates. Worst comparison means that a voter prefers a set of candidates $B$ over a set $C$ if he prefers the worst candidate in $B$ over the worst candidate in $C$. Best comparison is analogous but now comparing the best candidates of $B$ and $C$. A more refined notion is stochastic comparison: $B$ is preferred over $C$ if the lottery assigning equal probabilities over the candidates in $B$ stochastically dominates the lottery assigning equal probabilities over the candidates in $C$. Note, however, that the last preference extension is not complete: not each pair of sets can be compared this way.

Examples of manipulation

Here are some examples of manipulation of approval voting. There are six voters $(1,...,6)$ and four candidates $a$, $b$, $c$, $d$. We consider manipulation by voter 1 and under approval voting it is sufficient to know the total votes cast by the other voters.

• Assume that the votes from $2,...,6$ add up to $4$, $4$, $3$, $2$ for $a$, $b$, $c$, $d$, respectively. If voter 1 has preference $cab\!d$ (meaning that he prefers $c$ over $a$ over $b$ over $d$ and finds the first three acceptable), then truthful voting results in the winning set $\{a,b\}$. If 1 votes only for $a$ and $c$ then the winning set is $\{a\}$, which is better than $\{a,b\}$ both by worst and by stochastic comparison. If 1 votes only for $c$ then the winning set is $\{a,b,c\}$, which is better by best comparison.

• Now the votes cast by $2,...,6$ add up to $2$, $4$, $2$, $4$ for $a$, $b$, $c$, $d$, respectively and voter 1 has preference $ca\!b\!d$. Truthful voting results in $\{b,d\}$. Voting for $b$, $a$ and $c$ results in $\{b\}$, which is better by worst and stochastic comparison.

• Finally, the votes cast by $2,...,6$ add up to $3$, $4$, $2$, $2$ for $a$, $b$, $c$, $d$, respectively and voter 1 has preference $ca\!b\!d$. Truthful voting results in $\{b\}$. Voting for $a$ and $c$ results in $\{a,b\}$, which is better by best comparison.

Observe that in all these examples voter 1 still votes sincerely, even if he manipulates. This means that he still votes for a top ranked set of candidates. Nevertheless, he may sometimes not vote for a candidate even if he finds that candidate acceptable, or vote for a candidate even if he finds that candidate not acceptable.

The extent of potential manipulation

In Peters et al. (2009) the profiles of preferences in which some voter can manipulate, are characterized. The simplest way to obtain an idea about the extent of potential manipulation is to count the total number of manipulable profiles. In general this is combinatorially (too) complex. Using simulation, we have found that for the case of six voters and four candidates, as in the examples above, the percentages of manipulable profiles under worst, best, and stochastic comparison of winning sets are about 39%, 60%, and 80%, respectively. For ten voters and four candidates these numbers are 31%,
57%, and 73%. Some general trends can be distinguished, such as an increasing likelihood of manipulation from worst comparison to best comparison and stochastic comparison. Clearly, if the number of voters is large then manipulation is almost excluded since the probability that some voter is "pivotal," i.e., influences the winner, is small. Therefore, these percentages are relevant in particular for cases with relatively few voters, like elections of boards or councils of scientific communities. In spite of this, individual manipulation in general—not only for approval voting—is relevant also in large elections, like national elections for Parliament. This is so since individual voters may expect other voters with similar preferences to vote in the same way. For instance, in the Dutch elections for Parliament in January 2003, clearly the Social Democratic Party (PvdA) came out very strong as many (more) left-oriented people voted for it in order to decrease the probability of a conservative cabinet.

Approval voting and scoring rules

In a scoring rule there is a fixed (weakly) decreasing sequence of numbers that a voter has to assign to the candidates. The Borda rule in the first example above is a scoring rule. Closely related to approval voting is the k-approval scoring rule, according to which each voter assigns a 1 to exactly k candidates and a 0 to the remaining candidates. For k=1 this is the well known plurality rule: this rule is sometimes not manipulable at all, e.g. if there are only two voters, but it suffers from a serious drawback since it works to the disadvantage of candidates that are often high but second ranked. Consider, for instance, the position of a party like D66 in the Dutch political landscape.

There is some evidence that among all scoring rules, k-approval scoring rules with k>1 do well in terms of non-manipulability (Peters, Roy and Storcken (2008)). Specifically, if the number of voters is not too small then setting k equal to one half times the number of candidates seems to be minimally manipulable in terms of the total number of manipulable profiles. This matter is still under investigation. k-Approval scoring rules are less flexible than approval voting but seem to be also less manipulable. For instance, for our example with six voters and four candidates, the percentages of manipulable profiles under worst, best, and stochastic comparison are, respectively, 28%, 41%, and 52% for k=2, and 21%, 43%, and 64% for k=3.

Concluding remarks

Voting procedures are almost always manipulable. Approval voting offers a compromise between the possibility to report detailed information on one’s preference and non-manipulability, but the latter is violated to a lesser or larger extent if preferences are not dichotomous. Our research so far indicates that, in this respect, k-approval voting seems to offer a better compromise.

Of course, results like these should be considered in the right perspective. The number of manipulable preference profiles is just a crude measure for manipulability: it does not take the likelihood of such profiles within a certain population of voters into account, nor the seriousness of the consequences of manipulation. Also, even if possible, voters may abstain from manipulation simple because they do not have enough information about the votes of others to be able to successfully manipulate.

References


